

Eleven Euclidean Distances are Enough

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Introduction

The three distance theorem is a classic result in the study of distributions modulo 1, proved independently by several authors (see [5] and [6]) in the 1950s in response to a conjecture of Steinhaus. The theorem states that there are at most three distinct gaps between consecutive elements in the set of fractional parts of the first n multiples of any real number α . Formally, we have the following:

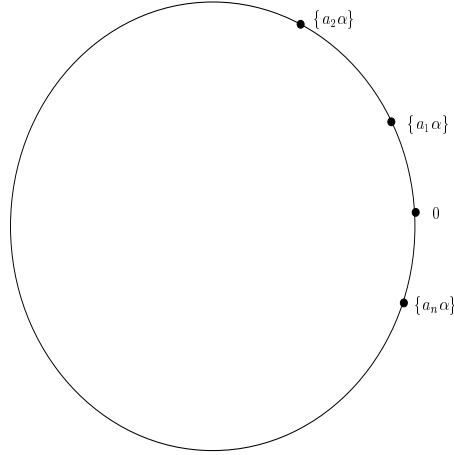
Theorem 1A Let α be any real number, and n a positive integer. Let (a_1, a_2, \dots, a_n) be the unique permutation of $\{1, 2, \dots, n\}$ such that

$$0 < \{a_1\alpha\} < \{a_2\alpha\} < \dots < \{a_n\alpha\} < 1$$

Define $g_\alpha(0) = \{a_1\alpha\}$ and $g_\alpha(n) = 1 - \{a_n\alpha\}$. For $1 \leq k \leq n-1$, let $g_\alpha(k) = \{a_{k+1}\alpha\} - \{a_k\alpha\}$. Define

$$S_\alpha(n) = \{g_\alpha(k) : 0 \leq k \leq n\}$$

Then $|S_\alpha(n)| \leq 3$.



Generalisations

Chung and Graham [3] generalised the three distance theorem as follows:

Theorem 2 Let $\alpha, \lambda_1, \lambda_2, \dots, \lambda_d$ be real numbers, and let n_1, n_2, \dots, n_d be positive integers. For $1 \leq i \leq d, 1 \leq k \leq n_i$, let $a_{i,k} = \{k\alpha + \lambda_i\}$, where $\{x\}$ denotes the fractional part of x . Then there are at most $3d$ distinct gaps between consecutive $a_{i,k}$.

Geelen and Simpson [4] established the following result, which was generalised by Chevallier [2] to higher dimensions:

Theorem 3 Let α and β be real numbers, and let n_1 and n_2 be positive integers. For $0 \leq k_1 < n_1, 0 \leq k_2 < n_2$, let $a_{k_1,k_2} = \{k_1\alpha + k_2\beta\}$. Then there are at most $n_1 + 3$ distinct gaps between consecutive a_{k_1,k_2} .

Chevallier [1] also obtained the following higher-dimensional analogue of the three-distance theorem for a certain subsequence of natural numbers.

Theorem 4 Let N be a best simultaneous approximation denominator with respect to the Euclidean norm of the d -tuple $(\alpha_1, \alpha_2, \dots, \alpha_d)$. Then there is a norm on R^d such that the Voronoi diagrams of the first N points of the sequence $(\{k\alpha_1\}, \{k\alpha_2\}, \dots, \{k\alpha_d\})$ with respect to this norm are of at most C_d different forms, where C_d is a constant that depends only on the dimension d .

A New Formulation

The purpose of this article is to show that the central tenet of the three-distance theorem, namely the finiteness of the set of minimal distances, can be generalised to higher dimensions under a suitable interpretation. We begin by rephrasing the theorem in a form that lends itself to the generalisation we seek.

We think of the three distance theorem as a statement about champions in a tournament. The players in the tournament are edges connecting $\{j\alpha\}$ and $\{k\alpha\}, 1 \leq j < k \leq n$, two edges play each other if and only if they overlap, and an edge loses only against edges of shorter length that it plays against. Defeated edges are allowed to play (and defeat) other overlapping edges. According to the three distance theorem, there are at most three distinct values

for the lengths of undefeated edges. Thus the theorem can be restated as follows:

Theorem 1B Let α be any real number, and n a positive integer. Define $d_\alpha(j, k) = ||\{k\alpha\} - \{j\alpha\}||$. Let $I_{j,k}$ be the “geodesic” joining $\{j\alpha\}$ with $\{k\alpha\}$, i.e., if $m_{j,k} = \min(\{j\alpha\}, \{k\alpha\})$ and $M_{j,k} = \max(\{j\alpha\}, \{k\alpha\})$, we define

$$I_{j,k} = \begin{cases} [m_{j,k}, M_{j,k}) & \text{if } M_{j,k} - m_{j,k} \leq 1/2 \\ [0, m_{j,k}) \cup [M_{j,k}, 1) & \text{otherwise} \end{cases}$$

Let $S_\alpha(n) = \{d_\alpha(j, k) : d_{\alpha,\beta}(p, q) < d_{\alpha,\beta}(j, k) \Rightarrow I_{p,q} \cap I_{j,k} = \emptyset\}$. Then $|S_\alpha(n)| \leq 3$.

We first prove a two-dimensional version of this theorem. We show that if the players are edges connecting $(\{j\alpha\}, \{j\beta\})$ and $(\{k\alpha\}, \{k\beta\})$ and two edges play each other if and only if their *projections along either co-ordinate axis* overlap, there are at most 11 distinct values for the lengths of undefeated edges. Numerical evidence suggests that the true value could be as small as 3.

Theorem 1 Let α and β be real numbers, and let n be a positive integer. Define $d_{\alpha,\beta}(j, k) = \sqrt{||(k-j)\alpha||^2 + ||(k-j)\beta||^2}$. Let $I_{j,k}^1$ and $I_{j,k}^2$ be the geodesics joining $\{j\alpha\}$ with $\{k\alpha\}$ and $\{j\beta\}$ with $\{k\beta\}$ respectively. Define

$$S_{\alpha,\beta}(n) = \{d_{\alpha,\beta}(j, k) : d_{\alpha,\beta}(p, q) < d_{\alpha,\beta}(j, k) \Rightarrow I_{p,q}^1 \cap I_{j,k}^1 = I_{p,q}^2 \cap I_{j,k}^2 = \emptyset\}$$

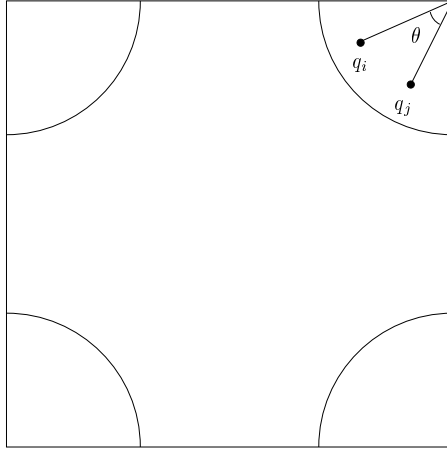
Then $|S_{\alpha,\beta}(n)| \leq 11$.

Proof We begin by classifying the denominators of simultaneous rational approximations to (α, β) . Let $[[x]] = \{x\} - 1/2$. We say that q is a denominator of type $(+, -)$ if $[[q\alpha]] \geq 0$ and $[[q\beta]] < 0$. Denominators of type $(-, +)$, $(+, +)$ and $(-, -)$ are defined analogously. Types $(+, +)$ and $(-, -)$ are said to be opposites to each other, as are types $(+, -)$ and $(-, +)$. We write $q_1 \parallel q_2$ if q_1 and q_2 are of the same type, $q_1 \perp q_2$ if they are of opposite type, and $q_1 \sim q_2$ if they are *not* of opposite type.

We define the *length* and the *angle* of an integer q with respect to α and β as $\ell(q) = d_{\alpha,\beta}(0, q)$, and $\theta(q) = \tan^{-1}([q\beta]/[q\alpha])$ respectively. Let Q_1 denote the least integer with the property that $\ell(Q_1) \leq \ell(q)$ for all q , $1 \leq q \leq n/2$. For $n/2 < q \leq n$, we say that q is *primary* if $\ell(q) < \ell(Q_1)$.

Lemma 1 If q is primary, $\ell(q)$ can take at most five distinct values.

Proof Consider four quarter-circles of radius $R = \ell(Q_1)$ centred at the four corners of the unit square. Suppose there exist $q_i, 1 \leq i \leq 7$ with $n/2 < q_1 < q_2 < \dots < q_7 \leq n$ and $\ell(q_i) < R$. Then there must be a pair $(i, j), 1 \leq i < j \leq 7$ such that $\theta \doteq |\theta(q_j) - \theta(q_i)| < \pi/3$. But then we have $\ell(q_j - q_i) < R$, yielding a contradiction, since $1 \leq q_j - q_i < n/2$.



Furthermore, the only way to have six primary q_i avoiding $\ell(q_j - q_i) < R$ is to arrange them along the vertices of a regular hexagon, leading to identical values of $\ell(q_i)$. It follows that $\ell(q)$ can take at most five distinct values if q is primary. ■

Consider the line $L_{a,b}$ joining $(\{a\alpha\}, \{a\beta\})$ and $(\{b\alpha\}, \{b\beta\})$, with $1 \leq a < b \leq n$. Let $q^* = b - a$. As we have seen, if q^* is primary, there are only five possible values for $\ell(q^*)$. Suppose q^* is not primary. We consider two cases.

CASE 1: $q^* \sim Q_1$ Note that one of $L_{a, a+Q_1}$ or $L_{b-Q_1, b}$ will be admissible, and will defeat $L_{a,b}$.

CASE 2: $q^* \perp Q_1$ Define $Q_1^\perp = \{q : 1 \leq q \leq n - Q_1, q \nparallel Q_1\}$. Note that Q_1^\perp is non-empty, since it contains q^* . Let Q_2 be the least integer in Q_1^\perp such that $\ell(q) \leq \ell(Q_2)$ for all $q \in Q_1^\perp$. We first prove the following lemma.

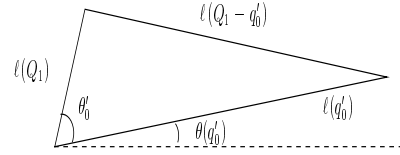
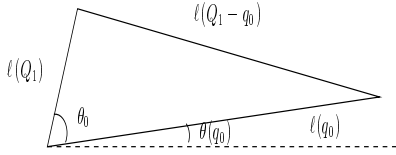
Lemma 2 There is at most one $q < Q_1$ satisfying $\ell(q) < \ell(Q_2)$.

Proof Suppose there are at least two such q . Let q_0 and q'_0 be the two smallest, with $q_0 < q'_0$. By the definition of Q_2 , $q_0 \parallel q'_0 \parallel Q_1$.

Let $x_1 = \|Q_1\alpha\|, x_2 = \|Q_1\beta\|, y_1 = \|q_0\alpha\|$ and $y_2 = \|q_0\beta\|$. Since $\ell(Q_1) \leq \ell(q_0)$, we have

$$\ell(Q_1 - q_0) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} \leq \sqrt{2(y_1^2 + y_2^2)} = \sqrt{2}\ell(q_0) < \sqrt{2}\ell(Q_2)$$

Since $Q_1 - q_0 \not\parallel Q_1$, it follows from the definition of Q_2 that $\ell(Q_1 - q_0) \geq \ell(Q_2) > \ell(q_0)$. Similarly, $\ell(Q_1 - q'_0) \geq \ell(Q_2) > \ell(q'_0)$. This requires $\theta_0 \doteq |\theta(Q_1) - \theta(q_0)| \geq \pi/3$ and $\theta'_0 \doteq |\theta(Q_1) - \theta(q'_0)| \geq \pi/3$.



It follows that $|\theta(q'_0) - \theta(q_0)| \leq \pi/6$, so $\ell(q'_0 - q_0) < \ell(Q_2)$. Since $\ell(q_0) < \ell(Q_2)$, we must have $q'_0 = 2q_0$. Since $q_0 \parallel q'_0$, we have,

$$\ell(q'_0) = 2\ell(q_0) \geq \sqrt{2}\ell(Q_1 - q_0) \geq \sqrt{2}\ell(Q_2)$$

contradicting our assumption about q'_0 . It follows that there is at most one $q < Q_1$ satisfying $\ell(q) < \ell(Q_2)$. ■

For $n - Q_1 < q \leq n$ and $q \perp Q_1$, we say that q is *secondary* if $\ell(q) < \ell(Q_2)$. If q^* is not secondary, one of $L_{a,a+Q_2}$ or $L_{a-Q_1,a}$ will be admissible, and will defeat $L_{a,b}$. We claim that if q^* is secondary, there are at most four distinct values that q^* can take.

Suppose not. Let $q_1 > q_2 > \dots > q_5 > n - Q_1$, with $\ell(q_i) < \ell(Q_2)$ and $q_i \perp Q_1$. Note that if $|\theta(q_i) - \theta(q_j)| \leq \pi/3$, we have $\ell(q_i - q_j) \leq \max(\ell(q_i), \ell(q_j)) < \ell(Q_2)$.

But given $q_1 \parallel q_2 \parallel \dots \parallel q_5$, it is easy to see that there exist (i_1, j_1) and (i_2, j_2) satisfying $q_{i_1} - q_{j_1} \neq q_{i_2} - q_{j_2}$ and

$$\max(|\theta(q_{i_1}) - \theta(q_{j_1})|, |\theta(q_{i_2}) - \theta(q_{j_2})|) \leq \pi/4 < \pi/3$$

Thus we have $\ell(q_{i_1} - q_{j_1}) < \ell(Q_2)$ and $\ell(q_{i_2} - q_{j_2}) < \ell(Q_2)$, contradicting Lemma 2. So there are at most four distinct values that $\ell(q^*)$ can take if $q^* \perp Q_1$ and q^* is secondary. It follows that at most eleven distinct gaps survive, completing the proof of the theorem. \blacksquare

Higher Dimensions

For higher dimensions, the above argument can be adapted to obtain similar results. We prove the following theorem which implies, in particular, that there are at most 290 distances in three dimensions.

Theorem 2 Let $\alpha \doteq (\alpha_1, \alpha_2, \dots, \alpha_m) \in \mathbb{R}^m$, and let n be a positive integer. Define

$$d_\alpha(j, k) = \sqrt{\sum_{i=1}^n \|(k - j)\alpha_i\|^2}$$

For $1 \leq r \leq m$, let $I_{j,k}^r$ denote the geodesic joining $\{j\alpha_r\}$ with $\{k\alpha_r\}$. Define

$$S_\alpha(n) = \{d_\alpha(j, k) : d_\alpha(p, q) < d_\alpha(j, k) \Rightarrow I_{p,q}^r \cap I_{j,k}^r = \emptyset \text{ for all } r.\}$$

Then

$$|S_\alpha(n)| \leq (\lceil \sqrt{m} \rceil^m)(\lceil \sqrt{m} \rceil^m + 2^m + 1) + 2$$

Proof Let $\llbracket x \rrbracket = \{x\} - 1/2$. As in the proof of Theorem 1, we assign, to each denominator q an m -tuple of signs. The i^{th} sign is positive if $\llbracket q\alpha_i \rrbracket \geq 0$ and negative otherwise.

The *length* of an integer q with respect to α is defined as $\ell(q) = d_\alpha(0, q)$. Let Q_1 denote the least integer with the property that $\ell(Q_1) \leq \ell(q)$ for all q , $1 \leq q \leq n/2$. For $n/2 < q \leq n$, we say that q is *primary* if $\ell(q) < \ell(Q_1)$.

Lemma 3 There are at most $(2\lceil \sqrt{m} \rceil)^m$ primary denominators in \mathbb{R}^m for any given α .

Proof If the number of distinct values q satisfying $\ell(q) < \ell(Q_1)$ exceeds $(2\lceil \sqrt{m} \rceil)^m$, at least $1 + \lceil \sqrt{m} \rceil^m$ of these values must be of the same type. By pigeonhole principle, there exists q_1 and q_2 with $\|(q_2 - q_1)\alpha_i\| < \ell(Q_1)/\sqrt{m}$ for all i , $1 \leq i \leq m$. It follows that $\ell(q_2 - q_1) < \ell(Q_1)$. But $q_2 - q_1 < n/2$, contradicting

the definition of Q_1 . Thus at most $(2\lceil\sqrt{m}\rceil)^m$ denominators can be primary. \blacksquare

Consider the line $L_{a,b}$ joining $(\{a\alpha\}, \{a\beta\})$ and $(\{b\alpha\}, \{b\beta\})$, with $1 \leq a < b \leq n$. Let $q^* = b - a$. As we have seen, if q^* is primary, there are only five possible values for $\ell(q^*)$. Suppose q^* is not primary. We consider two cases.

CASE 1: $q^* \sim Q_1$ Note that one of $L_{a,a+Q_1}$ or $L_{b-Q_1,b}$ will be admissible, and will defeat $L_{a,b}$.

CASE 2: $q^* \perp Q_1$ As in the planar case, define $Q_1^\perp = \{q : 1 \leq q \leq n - Q_1, q \not\parallel Q_1\}$, and let Q_2 be the least integer in Q_1^\perp such that $\ell(q) \leq \ell(Q_2)$ for all $q \in Q_1^\perp$. We prove the following analogue of Lemma 2.

Lemma 4 There are at most $\lceil\sqrt{2m}\rceil^m$ values of $q < Q_1$ satisfying $\ell(q) < \ell(Q_2)$.

Proof Suppose there are more. Then there exist q_0 and q'_0 with $\| (q'_0 - q_0)\alpha_i \| < \ell(Q_2)/\sqrt{2d}$ for all $i, 1 \leq i \leq m$. Thus $\ell(Q_2) > \sqrt{2}\ell(q''_0)$ where $q''_0 = q'_0 - q_0$. Note that $Q_1 - q''_0 \not\parallel Q_1$. Therefore $\ell(Q_2) < \ell(Q_1 - q''_0)$. We will now deduce a contradiction by showing that $\ell(Q_1 - q''_0) \leq \sqrt{2}\ell(q''_0)$.

Let $x_r = \|Q_1\alpha_r\|$ and $y_r = \|q''_0\alpha_r\|$. Since $\ell(Q_1) \leq \ell(q''_0)$, we have

$$\ell(Q_1 - q_0) = \sqrt{\sum_{r=1}^m (x_r - y_r)^2} \leq \sqrt{2 \sum_{r=1}^m y_r^2} = \sqrt{2}\ell(q''_0)$$

yielding the desired contradiction. It follows that there are at most $\lceil\sqrt{2m}\rceil^m$ values of $q < Q_1$ satisfying $\ell(q) < \ell(Q_2)$. \blacksquare

For $n - Q_1 < q \leq n$ and $q \perp Q_1$, we say that q is *secondary* if $\ell(q) < \ell(Q_2)$. If q^* is not secondary, one of $L_{a,a+Q_2}$ or $L_{a-Q_1,a}$ will be admissible, and will defeat $L_{a,b}$. We claim that if q^* is secondary, there are at most $(\lceil\sqrt{m}\rceil^m)(\lceil\sqrt{2m}\rceil^m + 1)$ distinct values that q^* can take.

Suppose not. Let $k = (\lceil\sqrt{m}\rceil^m)(\lceil\sqrt{2m}\rceil^m + 1) + 1$, and let

$$q_1 > q_2 > \cdots > q_k > n - Q_1$$

with $\ell(q_i) < \ell(Q_2)$. Then there exist $\lceil\sqrt{2m}\rceil^m + 1$ distinct denominators $q < Q_1$ with $\|q\alpha_r\| < \ell(Q_2)/\sqrt{m}$ for all $r, 1 \leq r \leq m$, thus satisfying $\ell(q) < \ell(Q_2)$ and

contradicting Lemma 4. So there are at most $(\lceil \sqrt{m} \rceil^m)(\lceil \sqrt{2m} \rceil^m + 1)$ distinct values that $\ell(q^*)$ can take if $q^* \perp Q_1$ and q^* is secondary. Accounting for Q_1 , Q_2 and primary denominators, we obtain the statement of the theorem. ■

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